KIRWAN MAP AND MODULI SPACE OF FLAT CONNECTIONS

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ABSTRACT. If K is a compact Lie group and $g \geq 2$ an integer, the space K^{2g} is endowed with the structure of a Hamiltonian space with a Lie group valued moment map Φ. Let β be in the centre of K. The reduction $\Phi^{-1}(\beta)/K$ is homeomorphic to a moduli space of flat connections. When K is simply connected, a direct consequence of a recent paper of Bott, Tolman and Weitsman is to give a set of generators for the K-equivariant cohomology of $\Phi^{-1}(\beta)$.

Another method to construct classes in $H_K^*(\Phi^{-1}(\beta))$ is by using the so called universal bundle. When the group is $\mathbf{SU}(n)$ and β is a generator of the centre, these last classes are known to also generate the equivariant cohomology of $\Phi^{-1}(\beta)$. The aim of this paper is to compare the classes constructed using the result of Bott, Tolman and Weitsman and the ones using the universal bundle.

In particular, I prove that the set of cohomology classes coming from the universal bundle is indeed a set of multiplicative generators for the cohomology of the moduli space. With $K = \mathbf{SU}(n)$, this is a new proof of the well-known construction of generators for the cohomology of the moduli space of semi-stable vector bundles with fixed determinant.

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INDEX OF NOTATION

$\mathbb{Z}$	Group of relative integers
$\mathbb{R}$	Field of reals
g	Integer bigger than 1
$\mathbf{F}$	Free group on $2g$ generators $x_1, \ldots, x_{2g}$
R	Element in $\mathbf{F}$ given by $\prod_{j=1}^{g}[x_{2j-1}, x_{2j}]$
$\mathbf{\Pi}$	Quotient of $\mathbf{F}$ by the relation $\prod_{j=1}^{g}[x_{2j-1}, x_{2j}]=1$
Σ	Closed Riemann surface of genus g
Σ_0	Σ with the interior of a disc removed
K	Simply connected compact Lie group
$Z(K)$	Centre of K
K_c	Quotient of K by its centre $Z(K)$

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1. Introduction

All cohomology will have coefficients in $\mathbb{R}$.

Let Σ be a closed compact Riemann surface of genus g. Let Σ_0 be a compact Riemann surface with boundary obtained from Σ by removing a small disc.

Let K be a compact simply connected Lie group. I denote by K_c the quotient of K by its centre $Z(K)$. In other words, K_c is the projectivisation of K. Let β be in the centre of K. Let K act on itself by conjugation and on K^{2g} by diagonal conjugation. The point β is fixed by K. Let $EK \longrightarrow BK$ be a classifying bundle for K. If M is a topological space acted on by K, the equivariant cohomology $H^*_K(M)$ of M is the singular cohomology of $M_K = M \times_K EK$. If M is a manifold and it is acted on smoothly by K, then $H^*_K(M)$ is also the cohomology of the Cartan-de Rham complex $\Omega^*_K(M)$.

Let Φ be the equivariant map

$$\begin{array}{ccc} K^{2g} & \longrightarrow & K \\ (X_1, \dots, X_{2g}) & \longmapsto & \prod_{i=1}^{g} [X_{2i-1}, X_{2i}]. \end{array}$$

Let $Y_\beta = \Phi^{-1}(\beta)$. The space $\Phi^{-1}(\beta)/K$ is homeomorphic to the moduli space of flat connections on the trivial bundle $K \times \Sigma_0 \longrightarrow \Sigma_0$ (all K principal bundles over Σ_0 are trivial) with prescribed holonomy β around the boundary S^1 of Σ_0.

Call κ the restriction map

$$\kappa : H^*_K(K^{2g}) \longrightarrow H^*_K(Y_\beta).$$

Let $b_1, \dots, b_r$ be primitive elements which generate $H^*(K)$ and let $c_1, \dots, c_r$ be their respective transgressions in $H^*(BK)$. One can extend $b_1, \dots, b_r$ to equivariant classes $b_1^K, \dots, b_r^K$ on K. Let $pr_i : K^{2g} \longrightarrow K$ be the projection on the i-th factor and $b_{j,i}^K$ the pull-back of b_j^K by pr_i.

Because Φ is a moment map on the quasi-Hamiltonian space K^{2g} (see [2]), the paper [5] gives a way of constructing equivariant classes $a_1, \dots, a_r$ on Y_β (see Equation (1) on page 7) such that $\{\kappa(c_j), \kappa(b_{j,i}^K), a_j,$

$j = 1, \ldots, r,\ i = 1, \ldots, 2g\}$ generate the K-equivariant cohomology of Y_β.

Also, one can construct a K-equivariant K_c-principal bundle over $\Sigma \times Y_\beta$: the so called universal bundle. The Künneth decomposition of the equivariant characteristic classes of this bundle allows me to construct classes $\{c'_j, b'_{j,i}, a'_j,\ j = 1, \ldots, r,\ i = 1, \ldots, 2g\}$ in $H^*_K(Y_\beta)$ (see Equation (2) on page 8). At least when K is the special unitary group and β is a generator of the centre of K, these classes are known to generate $H^*_K(Y_\beta)$ as a ring. They are the Atiyah-Bott-Biswas-Raghavendra classes[1].

The aim of this paper is to prove the following Theorem.

Theorem 3.2. — *Let K be a compact, connected and simply connected Lie group. Let β be in the centre of K such that K acts locally freely on Y_β. Let $a'_j, b'_{j,i}, c'_j, a_j, b^K_{j,i}, c_j$ be equivariant classes defined as above. The following relations hold*

$$\begin{array}{rcl} c'_j & = & \kappa(c_j) \\ b'_{j,i} & = & \kappa(b^K_{j,i}). \end{array}$$

Also, the classes $\{a_j\}$ depend on certain choices but one can choose them so that

$$a'_j = a_j.$$

This article is organised as followed. In Section 2, I recall what [5] says about the restriction map involved in the reduction of K^{2g} at β. I also show how to use this result to construct generators for the K-equivariant cohomology of Y_β. In Section 3, I describe the construction of equivariant characteristic classes on Y_β using a universal bundle over $\Sigma \times Y_\beta$. I also outline the proof of Theorem 3.2, leaving the most technical details for Section 4.

2. Generators for the equivariant cohomology of Y_β

Let $\mathbf{\Pi}$ be the fundamental group of Σ. Fix a presentation

$$\mathbf{\Pi} = \langle x_1, \ldots, x_{2g};\ \prod_{j=1}^{g} [x_{2j-1}, x_{2j}] = 1 \rangle.$$

[1]Atiyah and Bott [1] were the first ones to construct generators of the cohomology of the moduli space. Their classes depended on the choice of a universal bundle which was only defined up to the tensor product with a line bundle. The universal bundle of this article is actually the projectivisation of theirs. It is well-defined and so are the generators constructed from it. This was proved by Biswas and Raghavendra in [3]

Let $\mathbf{F}$ be the fundamental group of Σ_0. Fix a presentation

$$\mathbf{F} = \langle x_1, \ldots, x_{2g} \rangle$$

such that the map

$$\mathbf{F} \longrightarrow \mathbf{\Pi}$$

induced by the inclusion of Σ_0 in Σ, is given by the obvious map $x_k \longmapsto x_k$.

Let β be in the centre of K. Let Y_β be the subset of K^{2g} defined by

$$Y_\beta = \left\{ (X_1, \ldots, X_{2g}) \in K^{2g} \mid \prod_{j=1}^{g} [X_{2j-1}, X_{2j}] = \beta \right\}.$$

The group K acts on K^{2g} by conjugation. This action restricts to Y_β, since β is in the centre of K. The quotient $\mathcal{M}_\beta = Y_\beta / K$ can be identified with a moduli space of flat connections on the trivial principal bundle $K \times \Sigma_0 \longrightarrow \Sigma_0$ (because K is simply connected, all principal bundles are trivial over Σ_0), with holonomy β around the boundary $\partial\Sigma_0$. If K acts locally freely on Y_β, then the cohomology of $\mathcal{M}_\beta$ is isomorphic to the K-equivariant cohomology of Y_β (recall that cohomology is taken with coefficients in $\mathbb{R}$).

Let κ be the restriction map

$$\kappa : H_K^*(K^{2g}) \longrightarrow H_K^*(Y_\beta).$$

The space K^{2g} has the structure of a quasi-Hamiltonian space (see [2]) with moment map

$$\begin{array}{cccc} \phi : & K^{2g} & \longrightarrow & K \\ & (X_1 \ldots, X_{2g}) & \longmapsto & \prod_{j=1}^{g} [X_{2j-1}, X_{2j}]. \end{array}$$

The map ϕ is a submersion at any point of $Y_\beta = \phi^{-1}(\beta)$ if and only if K acts locally freely on Y_β. In this nice case, the space $\mathcal{M}_\beta$ is a compact symplectic orbifold since it is the regular reduction of K^{2g} at β.

Let $b_1, \ldots, b_r$ be primitive elements in $H^*(K)$ that generate the cohomology of K as a ring

$$H^*(K) = \bigwedge \left(\sum_{j=1}^{r} \mathbb{R} b_j \right).$$

Each b_j is of odd degree.

Let $c_1, \ldots, c_r$ be the transgressions in $H^*(BK)$ of respectively $b_1, \ldots, b_r$, thus $\deg c_j = \deg b_j + 1$ and

$$H^*(BK) = \mathbb{R}[c_1, \ldots, c_r].$$

For each K-principal bundle $G \longrightarrow X$, the classes $c_1, \ldots, c_r$ define characteristic classes $c_1(G), \ldots, c_r(G)$ in $H^*(X)$.

Proposition 2.1. *Let $EK \longrightarrow BK$ be the classifying bundle for K. The bundle $EK \times_K K^n \longrightarrow BK$ is cohomologically trivial as a ring, that is*

$$H^*_K(K^n) \simeq H^*(BK) \otimes H^*(K^n)$$

as rings.

This result is not new. I propose here a proof which is original as far as I know.

Also, because I want to state the principal results as soon as possible, I will make use in the following proof, as well as in others, of Lemma 3.5 even if I haven't proved it yet.

Proof. I will prove the case $n = 1$. The general case is similar, one only has to use Proposition 3.6 instead of Lemma 3.5 (or use the fact that if X has an equivariantly formal action of K then the diagonal action of K on X^n is also equivariantly formal).

Let me first prove that $H^*_K(K)$ and $H^*(BK) \otimes H^*(K)$ are isomorphic as $H^*(BK)$ modules. By the Leray-Hirsch Theorem, all I have to do is prove that the homomorphism

$$H^*_K(K) \longrightarrow H^*(K)$$

is surjective. Let b_j be one of the generators of $H^*(K)$. Let $D \longrightarrow S^1 \times K$ be the K-equivariant K-principal bundle of Lemma 3.5. As a K-principal bundle, its characteristic class $c_j(D)$ is $\mathrm{d}t \otimes b_j$ (Lemma 3.5). As an immediate corollary, I deduce that there exists b_j^K in $H^*_K(K)$ such that $c_j(D \times_K EK) = 1 \otimes c_j + \mathrm{d}t \otimes b_j^K$, where b_j^K restricts to b_j. Now, because the b_j's generate $H^*(K)$ as a ring, the homomorphism

$$H^*_K(K) \longrightarrow H^*(K)$$

is surjective and I have proved the existence of a $H^*(BK)$ module isomorphism between $H^*_K(K)$ and $H^*(BK) \otimes H^*(K)$. The existence of a ring isomorphism follows because the cohomology $H^*(K)$ of the fibre in $K \times_K EK \longrightarrow BK$ is generated by classes of odd degree (see [9]). $\square$

Remark 2.2. *Let s be the section*

$$\begin{array}{rccc} s: & BK & \longrightarrow & EK \times_K K \\ & [u] & \longmapsto & [u, 1]. \end{array}$$

Via $\mathrm{id} \times s : S^1 \times BK \longrightarrow S^1 \times K_K$*, the bundle $D_K \longrightarrow S^1 \times K_K$ pulls back to $S^1 \times EK \longrightarrow S^1 \times BK$. Hence $s^* b_j^K = 0$.*

Let n be an integer (I will need it to be either 1 or $2g$). Let $b^K_{j,i}$ be the pull-back, under the projection on the i-th factor $K^n \longrightarrow K$, of b^K_j.

The following Lemma is a consequence of Proposition 2.1.

Lemma 2.3. *The K-equivariant cohomology of K^n is isomorphic as a ring to*

$$H^*_K(K^n) \simeq \bigwedge(\sum \mathbb{R} b^K_{j,i}) \otimes \mathbb{R}[c_1, \ldots, c_r].$$

Before stating the result of Bott, Tolman and Weitsman for κ, I need to construct some classes in $H^*_K(Y_\beta)$.

Lemma 2.4. *For all j, the pull-back under ϕ^* of b^K_j is zero*

$$\phi^*(b^K_j) = 0.$$

In their article [5], the authors already gave this result. I give here a proof using Lemma 3.5.

Proof. Consider the diagram

$$\begin{array}{ccc} S^1 \times K^{2g} & \xrightarrow{m_1} & S^1 \times K \\ \Big\downarrow{\scriptstyle m_2} & & \\ \Sigma_0 \times K^{2g}, & & \end{array}$$

where the horizontal map m_1 is induced by the moment map $\phi : K^{2g} \longrightarrow K$ and the vertical map m_2 is given by the injection of S^1 as the boundary of Σ_0. The pull-back of D by m_1 and the pull-back of F by m_2 are isomorphic. Indeed they are both equivariantly isomorphic to the bundle

$$(\mathbb{R} \times K^{2g} \times K)/\mathbb{Z} \longrightarrow S^1 \times K^{2g}$$

where the group $\mathbb{Z}$ acts by

$$\begin{array}{ccc} \mathbb{Z} \times \mathbb{R} \times K^{2g} \times K & \longrightarrow & \mathbb{R} \times K^{2g} \times K \\ (\lambda, t, (X_1, \ldots, X_{2g}), k) & \longmapsto & (t + \lambda, (X_1, \ldots, X_{2g}), (\prod_{j=1}^g [X_{2j-1}, X_{2j}])^\lambda k). \end{array}$$

One then deduces the Lemma from Proposition 3.6 and Lemma 3.5 by noticing that the injection of S^1 as the boundary of Σ_0 induces the null map in degree 1 cohomology. ∎

Let $\bar{b_1}, \ldots, \bar{b_r}$ be equivariant forms representing respectively $b^K_1, \ldots, b^K_r$.

Remark 2.5. *Such forms will be chosen in Section 4 so that their pull-backs by the section s of Remark 2.2 vanish.*

The preceding Lemma implies there exist equivariant forms $\bar{a_1}, \ldots, \bar{a_r}$ on K^{2g} such that

$$\phi^*(\bar{b_j}) \;=\; \mathrm{d}\bar{a_j},\ \forall j. \tag{1}$$

The $\bar{a_1}, \ldots, \bar{a_r}$ are closed when restricted to Y_β. They define classes $a_1, \ldots, a_r$ in $H^*_K(Y_\beta)$.

Theorem 2.6 (Bott, Tolman and Weitsman)**.** *The equivariant cohomology $H^*_K(Y_\beta)$ is generated as a ring by the image of κ and the $a_1, \ldots, a_r$.*

The following Corollary is an immediate consequence.

Corollary 2.7. *The equivariant cohomology $H^*_K(Y_\beta)$ is generated as a ring by the classes $\kappa(c_j), \kappa(b^K_{j,i}), a_j$ for $j = 1, \ldots, r$ and $i = 1, \ldots, 2g$.*

3. Some classical classes in $H^*_K(Y_\beta)$

Let $\widetilde{\Sigma}$ be the universal cover of Σ. The group $\mathbf{\Pi}$ acts on $\widetilde{\Sigma}$. Since I have chosen a presentation for $\mathbf{F}$, there is a preferred isomorphism

$$\begin{array}{ccc} \mathrm{Hom}(\mathbf{F}, K) & \longrightarrow & K^{2g} \\ \rho & \longmapsto & (\rho(x_1), \ldots, \rho(x_{2g})). \end{array}$$

I will use this isomorphism to identify $\mathrm{Hom}(\mathbf{F}, K)$ and K^{2g}. Define an action of $\mathbf{\Pi}$ on $\widetilde{\Sigma} \times Y_\beta \times K_c$

$$\begin{array}{ccc} \mathbf{\Pi} \times (\widetilde{\Sigma} \times Y_\beta \times K_c) & \longrightarrow & \widetilde{\Sigma} \times Y_\beta \times K_c \\ (p, \sigma, \rho, k) & \longmapsto & (p\sigma, \rho, \rho(\tilde{p})k), \end{array}$$

where $\tilde{p}$ is any element in $\mathbf{F}$ lying above p. This action is well-defined and free. The quotient E, together with the projection $\pi : E \longrightarrow \Sigma \times Y_\beta$ define a K_c-principal bundle. The actions of K on Y_β by conjugation and on K_c by multiplication on the left, make E into a K-equivariant K_c-principal bundle over $\Sigma \times Y_\beta$. I call it the universal bundle (this bundle is also constructed in Jeffrey [7]).

Example 3.1. *For $K = \mathbf{SU}(n)$ and β a generator of $Z(\mathbf{SU}(n))$, this is the projectivisation of a universal bundle for the moduli space of semi-stable vector bundles of fixed rank and determinant (see* [1] *and* [9]*).*

Because K is compact and simply connected, I know that

$$H^*(BK) \simeq H^*(BK_c).$$

The classes $c_1, \ldots, c_r$ in $H^*(BK_c)$ define equivariant characteristic classes for the bundle E (recall that these are the same as the usual characteristic classes of the K-principal bundle $E \times_K EK \longrightarrow \Sigma \times Y_\beta \times_K EK$).

These classes live in $H^*_K(\Sigma\times Y_\beta)\simeq H^*(\Sigma)\otimes H^*_K(Y_\beta)$. Let $\alpha_1,\ldots,\alpha_{2g}$ be a basis of $H^*(\Sigma)$, dual to $x_1,\ldots,x_{2g}$ in $\mathbf{\Pi}=\pi_1(\Sigma)$. Let $\mathcal{V}$ be a volume form of volume 1 on Σ. Using the Künneth decomposition, write

$$c_j(E\times_K EK) \;=\; 1\otimes c'_j+\sum_{i=1}^{2g}\alpha_i\otimes b'_{j,i}+\mathcal{V}\otimes a'_j. \tag{2}$$

In Example 3.1, these classes are the known Atiyah-Bott – Biswas-Raghavendra generators of $H^*_K(Y_\beta)$. A natural question is: what is the relation between these classes and the ones of Corollary 2.7? The answer is given in the following Theorem.

Theorem 3.2. *Let K be compact, connected and simply connected Lie group. Let β be in the centre of K. Assume that β is a regular value of the moment map Φ. Let the $\{a'_j,b'_{j,i},c'_j,a_j,b^K_{j,i},c_j\}$ be equivariant classes defined as above. The following relations hold*

$$\begin{array}{rcl} c'_j &=& \kappa(c_j)\\ b'_{j,i} &=& \kappa(b^K_{j,i}). \end{array}$$

Also, the classes $\{a_j\}$ depend on certain choices but one can choose them so that

$$a'_j=a_j.$$

Proof. Fix j. I will start by proving the first part of the Theorem, that is I will compute the $\{c'_j,b'_{j,i}\}$.

Let $\widetilde{\Sigma}_0$ be the universal covering for Σ_0. Define an action of $\mathbf{F}=\pi_1(\Sigma_0)$ on $\widetilde{\Sigma}_0\times K^{2g}\times K$

$$\begin{array}{ccc} \mathbf{F}\times\widetilde{\Sigma}_0\times K^{2g}\times K & \longrightarrow & \widetilde{\Sigma}_0\times K^{2g}\times K\\ (p,\sigma,\rho,k) & \longmapsto & (p\sigma,\rho,\rho(p)k). \end{array}$$

This action is free and the quotient F is a K-principal bundle on $\Sigma_0\times K^{2g}$. The action of K by conjugation on K^{2g} and by multiplication on the left on K makes F into a K-equivariant K-principal bundle. The situation here is somehow confusing since there are two different actions of K on F. In order to clarify the situation, I will call the action that makes F a K-principal bundle the principal action and call the action that makes it K-equivariant the covering action ('covering' because it covers the action of K on K^{2g}).

Proposition 3.3. *The projectivisation of $F\mid_{\Sigma_0\times Y_\beta}$ is K-equivariantly isomorphic to $E\mid_{\Sigma_0\times Y_\beta}$.*

Proof. The proof is easy. □

Call D_n the restriction of F to the n-th circle in the wedge product times the n-th copy of K in K^{2g}. I identify the circle S^1 with the quotient $\mathbb{R}/\mathbb{Z}$. I denote by t the natural coordinate in $\mathbb{R}$, thus dt defines a volume form on S^1.

Lemma 3.4. *All the D_n's are isomorphic to the same bundle $D \longrightarrow S^1 \times K$. The total space D is $(\mathbb{R} \times K \times K)/\mathbb{Z}$ where the action of $\mathbb{Z}$ is defined by*

$$\begin{array}{ccc} \mathbb{Z} \times (\mathbb{R} \times K \times K) & \longrightarrow & \mathbb{R} \times K \times K \\ (\lambda, t, k_1, k_2) & \longmapsto & (t+\lambda, k_1, k_1^\lambda k_2) \end{array}$$

and the projection is

$$\begin{array}{ccc} (\mathbb{R} \times K \times K)/\mathbb{Z} & \longrightarrow & S^1 \times K \\ \{t, k_1, k_2\} & \longmapsto & (\{t\}, k_1). \end{array}$$

Proof. Fix an integer n between 1 and $2g$ and let $x = x_n$ be the n-th generator of $\mathbf{F}$.
Recall that F was defined as a quotient

$$F = (\widetilde{\Sigma}_0 \times K^{2g} \times K)/\pi_1(\Sigma_0).$$

Let h be the group homomorphism

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \pi_1(\Sigma_0) \\ \lambda & \longmapsto & x^\lambda. \end{array}$$

This morphism makes $\mathbb{Z}$ into a subgroup of $\pi_1(\Sigma_0)$. Choose a point $\widetilde{\sigma_0}$ in Σ_0 lying above the base point σ_0 of Σ_0. Choose a loop $S^1 \longrightarrow \Sigma_0$ representing x. This loop lifts to a map $\chi : \mathbb{R} \longrightarrow \widetilde{\Sigma}_0$ such that $\chi(0) = \widetilde{\sigma_0}$ and for λ an integer $\chi(t+\lambda) = x^\lambda \cdot \chi(t)$.

The covering action of K (the one that covers the action of K on $S^1 \times K$) on D_n is induced by the conjugation on the first factor K in $\mathbb{R} \times K \times K$ and by multiplication on the left on the second factor K.

Let $\iota : K \longrightarrow K^{2g}$ be the injection of the n-th factor. Define a map

$$\begin{array}{ccc} (\mathbb{R} \times K \times K)/\mathbb{Z} & \longrightarrow & (\widetilde{\Sigma}_0 \times K^{2g} \times K)/\pi_1(\Sigma_0) \\ [t, X, k] & \longmapsto & [\chi(t), \iota(X), k]. \end{array}$$

This map is well-defined since it sends the translated $[t+\lambda, X, X^\lambda k]$ of $[t, X, k]$ by λ to

$$\begin{array}{rl} & [\chi(t+\lambda), \iota(X), X^\lambda k] \\ = & [x_n^\lambda \cdot \chi(t), \iota(X), X^\lambda k] \\ = & [\chi(t), \iota(X), k]. \end{array}$$

It is also K-equivariant because, for ℓ in K, it sends $[t, \mathrm{Ad}_\ell X, \ell k]$ to

$$\begin{array}{rl} & [\chi(t), \iota(Ad_\ell X), \ell k] \\ = & [\chi(t), Ad_\ell \circ \iota(X), \ell k] \\ = & \ell \cdot [\chi(t), \iota(X), k]. \end{array}$$

The Lemma follows. □

Lemma 3.5. *The characteristic classes of D are*

$$c_j(D) = dt \otimes b_j.$$

Proof. The proof is to be found in the next Section. □

The previous Lemma is actually very useful since it can be used to prove Proposition 3.6 which leads to Theorem 3.2, and also Proposition 2.1 and Lemma 2.4.

Now I wish to compute the equivariant characteristic classes of F. The cohomologies of Σ and Σ_0 are the same in degree 0 and 1 and Σ_0 has no cohomology in degree 2. I use the same basis $\alpha_1, \ldots, \alpha_{2g}$ for $H^1(\Sigma)$ and $H^1(\Sigma_0)$.

Proposition 3.6. *Using the Künneth decomposition, the equivariant characteristic classes of F are*

$$c_j(F \times_K EK) = 1 \otimes c_j + \sum_{i=1}^{2g} \alpha_i \otimes b^K_{j,i}.$$

Proof. The surface Σ_0 is homotopic to a wedge product of $2g$ circles. As is easily seen, Proposition 3.6 will follow from the computation of the characteristic classes of the restriction of F to the n-th circle in the wedge product times the m-th copy of K in K^{2g}, that is D_n. When $n \neq m$, the restriction of F is trivial. Thus, I only have to do the computation for $n = m$. This was done in Lemma 3.5. □

The first part of the Theorem is a consequence of Proposition 3.3 and Proposition 3.6. Indeed, by Proposition 3.3

$$c_j(E \times_K EK) \mid_{\Sigma_0 \times Y_\beta} = c_j(F \times_K EK) \mid_{\Sigma_0 \times Y_\beta} .$$

But

$$c_j(E \times_K EK) \mid_{\Sigma_0 \times Y_\beta} = 1 \otimes c'_j + \sum_{i=1}^{2g} \alpha_i \otimes b'_{j,i} \tag{3}$$

and by Proposition 3.6

$$
\begin{aligned}
c_j(F \times_K EK) \mid \Sigma_0 \times Y_\beta &= (\mathrm{id}^* \times \kappa) c_j(F \times_K EK) \\
&= 1 \otimes \kappa(c_j) + \sum_{i=1}^{2g} \alpha_i \otimes \kappa(b^K_{j,i}).
\end{aligned} \tag{4}
$$

The first part of the Theorem follows by comparing Equation (3) and Equation (4).

The second part of Theorem 3.2 is a consequence of the following Lemma and of the definition of the classes $a_1, \ldots, a_r$.

Lemma 3.7. *One can choose representatives $\bar{b}^K_j$ and $\bar{a}'_j$ of respectively b^K_j and a'_j such that $\bar{a}'_j$ extends to an equivariant class on K^{2g} whose differential is $\Phi^*(\bar{b}^K_j)$.*

The proof of this Lemma, together with the proof of Lemma 3.5, is given in the next Section.

This finishes the proof of Theorem 3.2. □

Remark 3.8. *If the group K is a torus T, then it is not simply connected and one can not apply Theorem 3.2 to find generators of the cohomology of the moduli space. Nevertheless, in this case the moduli space is either T^{2g} if $\beta = 1$ or is empty otherwise. Its cohomology is thus very simple. In general, a connected compact Lie group is the quotient by a finite group of a product of a simply connected group and a torus. Combining what I have just said about tori with Theorem 3.2, one can deduce generators for the cohomology of the moduli space.*

4. Proof of Lemma 3.7 and Lemma 3.5

The aim of this section is to prove Lemma 3.5 and Lemma 3.7.

The proof of Lemma 3.5 goes as follows: in a similar way to Jeffrey[7], I construct explicitly a map from $S^1 \times K$ to the fat realisation of BK as a simplicial manifold such that the pull-back of EK under this map is the principal bundle D_n. I then use a result of Shulman to prove Lemma 3.5.

I will start by introducing the simplicial realisations of EG and BG for any group G (see [10]). Let

$$
\Delta^m = \left\{ (t_0, \ldots, t_m) \in [0,1]^{m+1} \mid \sum_{i=0}^{m} t_i = 1 \right\}
$$

be the standard m-simplex. The simplicial realisation $\overline{\mathrm{NG}}$, respectively NG, of EG, respectively BG, is given by the contravariant functor

$$\{\Delta^m,\, m \in \mathbb{N}\} \longrightarrow \{\text{category of smooth manifolds}\}$$

which to each Δ_m associates $\overline{\mathrm{NG}}(m) = G^{m+1}$, respectively $\mathrm{NG}(m) = G^m$. For i between 0 and m, the map $\bar{\epsilon_i} : \overline{\mathrm{NG}}(m) \longrightarrow \overline{\mathrm{NG}}(m-1)$, respectively $\epsilon_i : \mathrm{NG}(m) \longrightarrow \mathrm{NG}(m-1)$, corresponding to the i-th face map $\epsilon^i : \Delta^{m-1} \longrightarrow \Delta^m$ is given by the omission of the i-th term

$$\bar{\epsilon_i}(k_0, \ldots, k_m) = (k_0, \ldots, \check{k}_i, \ldots, k_m),$$

respectively

$$(5)\epsilon_i(k_1, \ldots, k_m) = \begin{cases} (k_2, \ldots, k_m) & i = 0 \\ (k_1, k_2, \ldots, k_i k_{i+1}, \ldots, k_m) & i = 1, \ldots, m-1 \\ (k_1, \ldots, k_{m-1}) & i = m. \end{cases}$$

The projection $\overline{\mathrm{NG}}(m) \longrightarrow \mathrm{NG}(m)$ is the map

$$q_m(k_0, \ldots, k_m) = (k_1 k_2^{-1}, \ldots, k_{m-1} k_m^{-1}).$$

I then take EG, respectively BG, to be the fat realisation of $\overline{\mathrm{NG}}$, resp. NG, that is EG is the quotient of $\bigsqcup_m \Delta^m \times \overline{\mathrm{NG}}(m)$ by the relation

$$(\epsilon^i \times \mathrm{id})(t, x) \sim (id \times \bar{\epsilon_i})(tx)$$

for (t, x) in $\Delta^{m-1} \times \overline{\mathrm{NG}}(m)$ and BG is the quotient of $\bigsqcup_m \Delta^m \times \mathrm{NG}(m)$ by the relation

$$(\epsilon^i \times \mathrm{id})(t, x) \sim (id \times \epsilon_i)(tx)$$

for (t, x) in $\Delta^{m-1} \times \mathrm{NG}(m)$. The projection $q : EG \longrightarrow BG$ is induced by the maps q_m. The action of G on EG is induced by the diagonal multiplication of G on the right on $\overline{\mathrm{NG}}(m) = G^{m+1}$ for each m. In fact, there is a second action of G on EG, the covering action, which descends to an action of G on BG. This action on EG is induced by the diagonal multiplication on the left of G on $\overline{\mathrm{NG}}(m) = G^{m+1}$, it descends to an action on BG induced by the diagonal conjugation on $\mathrm{NG}(m) = G^m$.

Notice that

$$\begin{aligned} S^1 &= B\mathbb{Z} \\ \Sigma &= B\mathbf{\Pi} \\ \Sigma_0 &= B\mathbf{F}. \end{aligned}$$

In the simplicial models, the injection of Σ_0 in Σ is given by the projections

$$\mathbf{F}^n \longrightarrow \mathbf{\Pi}^n.$$

The surface Σ_0 is homotopic to the wedge product of $2g$ circles. The inclusion of the i-th circle $S^1 \longrightarrow \Sigma_0$ is given by

$$\begin{array}{ccc} \mathbb{Z}^n & \longrightarrow & \mathbf{F}^n \\ \lambda & \longmapsto & (x_i^\lambda, \ldots, x_i^\lambda). \end{array}$$

Let $R = \prod_{j=1}^{g} [x_{2j-1}, x_{2j}]$ be in $\mathbf{F}$. The inclusion of S^1 as the boundary of Σ_0 is the realisation of the simplicial map

$$\begin{array}{ccc} \mathbb{Z}^n & \longrightarrow & \mathbf{F}^n \\ \lambda & \longmapsto & (R^\lambda, \ldots, R^\lambda). \end{array}$$

The classifying map $f : S^1 \times K \longrightarrow BK$ for the bundle D is the realisation of the simplicial map

$$\begin{array}{cccc} f : & \mathbb{Z}^n \times K & \longrightarrow & K^n \\ & (k, \lambda_1, \ldots, \lambda_n) & \longmapsto & (k^{\lambda_1}, \ldots, k^{\lambda_n}). \end{array}$$

The homology of S^1 is the Eilenberg-Mac Lane group homology $H_*(\mathbb{Z})$ of $\mathbb{Z}$ (see [8]). Denote by x a generator of $\mathbb{Z}$. A generator of $H_1(\mathbb{Z})$ is given by x.

Since the centre of K is finite, on the one hand K and K_c, and on the other hand BK and BK_c have naturally isomorphic cohomologies. In addition, every K-principal bundle defines a K_c-principal bundle in a natural way: one just has to compose the classifying map of the K-principal bundle with the projection $BK \longrightarrow BK_c$. The simplicial version of this last projection is given by the maps $K^n \longrightarrow K_c^n$. Moreover, the characteristic classes of the induced K_c-principal bundle are the same as the one of the K-principal bundle.

The cohomology of BK_c is the cohomology of a double complex $\Omega^{p,q} = \Omega^p(\mathrm{NK}_c(q))$ with two differentials: one is the usual differential of forms $\mathrm{d} : \Omega^p(\mathrm{NK}_c(q)) \longrightarrow \Omega^{p+1}(\mathrm{NK}_c(q))$ and the other is

$$\begin{array}{c} \delta : \Omega^p(\mathrm{NK}_c(q)) \longrightarrow \Omega^p(\mathrm{NK}_c(q+1)) \\ \delta = \sum_{i=0}^{q+1} (-1)^i \varepsilon_i^* \end{array}$$

Recall that the maps ε_i were defined in (4). The two differentials d and δ commute and the total differential is $\mathrm{d} + (-1)^p \delta$ on $\Omega^{p,q}$. Shulman [11] (see also Bott, Shulman and Stasheff [4]) proved that each class c_j in $H^{2r_j}(BK_c)$ can be represented by a form $c_{j,1} + \cdots + c_{j,r}$ in $\Omega^{*,*}$, with $c_{j,k} \in \Omega^{2r-k,k}$. That this form is closed means that

$$\begin{array}{rcl} \mathrm{d}c_{j,1} & = & 0 \\ \delta c_{j,k} & = & (-1)^k \mathrm{d}c_{j,k+1} \text{ for } 1 \leq j < r_j \\ \delta c_{j,r} & = & 0. \end{array}$$

Also, the closed form $c_{j,1} \in \Omega^{2r_j-1}(K_c)$ is a representative of b_j.

Everything in the above paragraph can be done equivariantly. One just has to replace cohomology by equivariant cohomology, the complex $\Omega^{p,q} = \Omega^p(\mathrm{NK}_c(q))$ by $\Omega_K^{p,q} = \Omega_K^p(\mathrm{NK}_c(q))$, the differential d by d_K, the forms c_j by their respective extensions c_j^K in $H_K^{2r_j}(BK_c)$, and so on. Hence it is clear that the pairing $\langle x, f^*c_j\rangle$ is b_j^K. This together with the fact that the restriction of D to S^1 is trivial proves Lemma 3.5.

I chose $c_{j,1}^K$ as a representative $\bar{b}_j^K$ of b_j^K. The section

$$\begin{array}{cccc} s: & BK & \longrightarrow & EK \times_K K \\ & [u] & \longmapsto & [u,1] \end{array}$$

is obtained from the inclusion of the identity 1 into K by applying the functor, from K-spaces to spaces,

$$X \longmapsto X \times_K EK.$$

Because the inclusion map $\{1\} \longrightarrow K$ is constant and because $c_{j,1}^K$ is of odd degree, it follows that $s^*b_j^K = s^*c_{j,1}^K = 0$. This proves Remark 2.5.

The equivariant forms $c_{j,1}^K$ and $c_{j,2}^K$ satisfy $\mathrm{d}_K c_{j,2}^K = -\delta c_{j,1}^K$. A classifying map for the K-equivariant K_c-principal bundle E is the realisation of the equivariant simplicial map

$$\begin{array}{cccc} \Psi: & Y_\beta \times \Pi^n & \longrightarrow & K_c^n \\ & (\rho, p_1, \ldots, p_n) & \longmapsto & (\rho(p_1), \ldots, \rho(p_n)). \end{array}$$

The homology of Σ is the Eilenberg-Mac Lane group homology $H_*(\mathbf{\Pi})$. A generator of $H_2(\mathbf{\Pi})$ is (see [6, Proposition 3.9])

$$c = \sum_{i=1}^{2g} \frac{\partial R}{\partial x_i} \otimes x_i,$$

where $\frac{\partial R}{\partial x_i}$ is the Fox free differential calculus (see [6, Section 3]). In fact,

$$\frac{\partial R}{\partial x_i} = \gamma_j^0 - \gamma_i^1,$$

where $\gamma_i^\tau \in \mathbf{F}$ is

$$\begin{array}{rcl} \gamma_{2i-1}^0 & = & \prod_{l=1}^{i-1}[x_{2l-1}, x_{2l}], \\ \gamma_{2i-1}^1 & = & \gamma_{2i-1}^0 x_{2i-1} x_{2i} x_{2i-1}^{-1}, \\ \gamma_{2i}^0 & = & \gamma_{2i-1}^0 x_{2i-1}, \\ \gamma_{2i}^1 & = & \gamma_{2i-1}^0 [x_{2i-1}, x_{2i}]. \end{array}$$

A representative $\bar{a}'_j$ of a'_j is given by

$$\begin{array}{rcl} \bar{a}'_j & = & \langle \sum_{i=1}^{2g} \frac{\partial R}{\partial x_i} \otimes x_i, \Psi^* c_{j,2}^K \rangle \\ & = & \sum_{i=1}^{2g} \sum_{\tau=0,1} (-1)^\tau \langle \gamma_i^\tau \otimes x_i, \Psi^* c_{j,2}^K \rangle \\ & = & \sum_{i=1}^{2g} \sum_{\tau=0,1} (-1)^\tau (\mathrm{ev}_{\gamma_i^\tau} \times \mathrm{ev}_{x_i})^* c_{j,2}^K, \end{array}$$

where, for $p \in \mathbf{F}$, the evaluation map is

$$\begin{array}{rccc} ev_p : & \mathrm{Hom}(\mathbf{F}, K_c) & \longrightarrow & K_c \\ & \rho & \longmapsto & \rho(p). \end{array}$$

The last line of the above computation of $\bar{a}'_j$ clearly shows that $\bar{a}'_j$ extends to a form on K^{2g}. The differential of this form is

$$\begin{aligned} \mathrm{d}\bar{a}'_j &= \sum_{i=1}^{2g} \sum_{\tau=0,1} (-1)^\tau (\mathrm{ev}_{\gamma_i^\tau} \times \mathrm{ev}_{x_i})^* \mathrm{d}c^K_{j,2} \\ &= -\sum_{i=1}^{2g} \sum_{\tau=0,1} (-1)^\tau (\mathrm{ev}_{\gamma_i^\tau} \times \mathrm{ev}_{x_i})^* \delta c^K_{j,1} \\ &= -\sum_{i=1}^{2g} \sum_{\tau=0,1} (-1)^\tau (\mathrm{ev}_{\gamma_i^\tau} \times \mathrm{ev}_{x_i})^* \sum_{l=0}^{2} (-1)^l \varepsilon_l^* c^K_{j,1} \\ &= -\sum_{i=1}^{2g} \sum_{\tau=0,1} \sum_{l=0}^{2} (-1)^{\tau+l} (\varepsilon_l \circ (\mathrm{ev}_{\gamma_i^\tau} \times \mathrm{ev}_{x_i}))^* c^K_{j,1} \\ &= \sum_{i=1}^{2g} \sum_{\tau=0,1} \sum_{l=0}^{2} (-1)^{1+\tau+l} \mathrm{ev}^*_{z^\tau_{i,l}} c^K_{j,1}, \end{aligned} \tag{6}$$

where

$$z^\tau_{i,0} = x_i,\ z^\tau_{i,1} = \gamma_i^\tau x_i \text{ and } z^\tau_{i,2} = \gamma_i^\tau.$$

Because

$$\begin{aligned} \mathrm{ev}^*_{z^0_{i,0}} &= \mathrm{ev}^*_{z^1_{i,0}} \\ \mathrm{ev}^*_{z^1_{2i,1}} &= \mathrm{ev}^*_{z^1_{2i-1,2}} \\ \mathrm{ev}^*_{z^0_{2i-1,1}} &= \mathrm{ev}^*_{z^0_{2i,2}} \\ \mathrm{ev}^*_{z^0_{2i,1}} &= \mathrm{ev}^*_{z^1_{2i-1,1}} \\ \mathrm{ev}^*_{z^0_{2i+1,2}} &= \mathrm{ev}^*_{z^1_{2i,2}} \end{aligned}$$

many terms cancel each other in the expression (6) above. I am left with

$$\mathrm{d}\bar{a}'_j = \mathrm{ev}^*_{z^1_{2g,2}} c^K_{j,1} - ev^*_{z^0_{1,2}} c^K_{j,1}.$$

But $z^0_{1,2} = 1$ the identity of $\mathbf{F}$ and hence $ev_{z^0_{1,2}}$ is a constant map, whereas $z^1_{2g,2} = \prod_{j=1}^g [x_{2j-1}, x_{2j}]$ and hence $\mathrm{ev}^*_{z^1_{2g,2}} = \Phi^*$. Since $c^K_{j,1}$ is a representative of b^K_j, I deduce that on K^{2g}

$$\mathrm{d}\bar{a}'_j = \Phi^* \bar{b}^K_j.$$

This proves Lemma 3.7.

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